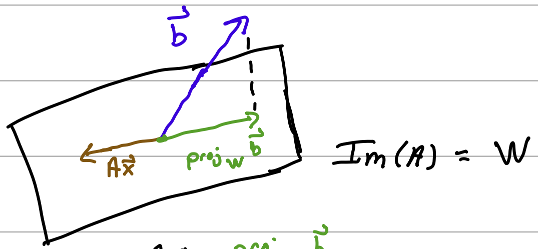
Lecture 26

Def: $\vec{x}^{**} \in \mathbb{R}^n$ is a least squares solution of an $m \times n$ linear system $A\vec{x} = \vec{b}$ if the distance from $A\vec{x}$ to $\vec{b}$ is minimal.

i.e. $\|A\vec{x}^* - \vec{b}\| \leq \|A\vec{x} - \vec{b}\| \quad \forall \vec{x} \in \mathbb{R}^n$

Rmk: If $\vec{x}$ solves $A\vec{x} = \vec{b}$ then $\|A\vec{x} - \vec{b}\| = 0$ so actual solutions are least squares solutions.



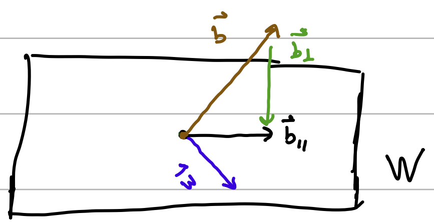
$Ax^* = \text{proj}_W \vec{b}$

Prop: Given a subspace $W \subset \mathbb{R}^n$ and a vector $\vec{b} \in \mathbb{R}^m$, we have $\forall \vec{w} \in W$ that

$\|\vec{w} - \vec{b}\| \geq \|\text{proj}_W \vec{b} - \vec{b}\|$

Pf:

$$\|\vec{w} - \vec{b}\|^2 = \left\|(\vec{w} - \text{proj}_W \vec{b}) + (\text{proj}_W \vec{b} - \vec{b})\right\|^2$$



Obs: $A\vec{x} \in W := Im A$ = spans of cols of A

- So by proposition $\|A\vec{x} - \vec{b}\|$ is minimal if $A\vec{x} = \text{proj}_W \vec{b}$

  i.e. if $(A\vec{x} - \vec{b}) \perp W$

  i.e. if $(A\vec{x} - \vec{b}) \perp$ every col of A

$\vec{b}_\perp = (A\vec{x} - \vec{b})$

i.e. $A^T \vec{b}_T = 0$

$$\begin{bmatrix} (\mathrm{col}_1 A)^T \\ \vdots \\ (\mathrm{col}_n A)^T \end{bmatrix} \vec{b}_\perp = \begin{bmatrix} (\mathrm{col}_1 A) \cdot \vec{b}_\perp \\ \vdots \\ (\mathrm{col}_n A) \cdot \vec{b}_\perp \end{bmatrix}$$

- So we want $A^T(A\vec{x} - \vec{b}) = 0$

$$\rightarrow A^T A \vec{x} = A^T b \rightarrow \text{"Normal Equation"}$$

<u>Theorem</u>: $\vec{x}^* \in \mathbb{R}^n$ is a least squares (LS) solution of an $m \times n$ linear system $A\vec{x} \in \vec{b}$ i.f.f. $\vec{x}^*$ solves the associated system $A^T A \vec{x} = A^T \vec{b}$

$\boxed{\text{Ex}}$

$$\left[\begin{array}{cc|c} 1 & 1 & 4 \\ 2 & -1 & 0 \\ 3 & -1 & 1 \end{array}\right] = [A \mid \vec{b}]$$

Get the least squares solution:

↳ multiply by A^T by A, and A^T by $\vec{b}$.

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & -1 & -1 \end{bmatrix} \left[\begin{array}{cc|c} 1 & 1 & 4 \\ 2 & -1 & 0 \\ 3 & -1 & 1 \end{array}\right] = \left[\begin{array}{cc|c} 14 & -4 & 7 \\ -4 & 3 & 3 \end{array}\right] \rightarrow \underbrace{\vec{x} = \begin{pmatrix} 33/26 \\ 35/13 \end{pmatrix}}_{\text{This is the point closest to the solution.}}$$

$\boxed{\text{Ex}}$ What line in $y = c_0 + c_1 x$ "best fits" the points $(1,0)$, $(3,2)$, $(4,3)$, $(1,-1)$

<u>Solution:</u> We want:

$$\left.\begin{aligned} c_0 + (1)c_1 &= 0 \\ c_0 + (2)c_1 &= 2 \\ c_0 + (4)c_1 &= 3 \\ c_0 + (-1)c_1 &= -1 \end{aligned}\right\} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 3 & 2 \\ 1 & 4 & 3 \\ 1 & -1 & -1 \end{array}\right]$$

Use least squares solution:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 3 & 4 & -1 \end{bmatrix} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 1 & 3 & 2 \\ 1 & 4 & 3 \\ 1 & -1 & -1 \end{array}\right] = \left[\begin{array}{cc|c} 4 & 7 & 4 \\ 7 & 27 & 19 \end{array}\right] \rightarrow \begin{bmatrix} c_0 \\ c_1 \end{bmatrix} = \frac{1}{59}\begin{pmatrix} -25 \\ 48 \end{pmatrix}$$

So the best fit line is $y = c_0 + c_1(x)$

$$y = (-0.424) + (8.14)x$$